Theory of a contact transport in a gas-like medium

1991 J. Phys. A: Math. Gen. 24 651

(http://iopscience.iop.org/0305-4470/24/3/024)

J. Phys. A: Math. Gen. 24 (1991) 651-665. Printed in the UK

Theory of a contact transport in a gas-like medium

A V Gurevich† and G M Milikh‡

Department of Physics and Astronomy, Clemson University, Clemson, South Carolina 29634-1911, USA

Received 25 May 1990, in final form 6 November 1990

Abstract. We discuss in this paper the structure of moving fronts, wave motions and self-excitation of oscillations from the point of view of the contact transport in a gas-like medium. The generalized equation of a contact transport in a system of free moving elements located far apart from each other is formulated. The parameters of contact, properties and distribution of elements are in general some functions of time. Several important solutions of these equations are obtained and analysed. Applications to other related problems such as spread of a contagious disease and percolation are briefly discussed.

1. Introduction

The transport processes in gases and plasma have been widely discussed. The approach is based on the solution of equations of transport for different macroscopic parameters: density, momentum, temperature and charge. These equations are found in relation to collisions of atoms (or molecules) moving in an external field. Sometimes, as with the chemical kinetics of polyatomic molecules, freely moving particles may interact in a complicated way. In fact, the properties of such particles may vary in time thus affecting their interactions. This allows another possible interpretation of the above transport processes in gas-like systems in terms of mathematical theory of contagious diseases. Because of this analogy, it is convenient to consider the above particles as elements. In the language of the deterministic epidemic model the freely moving objects (which we call infectious) are contacting the elements of background (which we call susceptibles) [1]. Different methods have been applied for the treatment of the above problems. In the mathematical theory of epidemics based on time varying processes the simplified form of the integral of contact transfer has been used [2-5]. The processes of the propagation in space have been studied numerically based on computer models described in [6, 7]. The transport phenomena in strongly non-uniform systems have been studied using the methods of directed percolation theory [8-10]. The ideas of the mean-field renormalization group have also been applied to the contact process [11].

The goal of the present work is to develop and to analyse generalized equations of a contact transport inside a non-uniform system of freely moving elements located far apart from each other. The parameters of contact, properties and distribution of elements may be some functions of time. A few important solutions of these equations will be discussed.

† Permanent address: Lebedev Institute of Physics, Moscow, USSR.
‡ Present address: Astronomy Program, University of Maryland, College Park, MD 20742, USA.

0305-4470/91/030651+15$03.50

This paper is organized as follows. The main features of contact transport inside a gas-like media are discussed in section 2. The equations of contact transport are obtained in section 3 while section 4 is devoted to the investigation of the possibility of propagation of nonlinear waves in the medium just described. It is found that two steady states can exist in such a system. The stability of these states is investigated in section 5 along with the possibility of the self-excitation of oscillations. The transport processes in strongly non-uniform systems are discussed in section 6 while section 7 is devoted to a brief discussion.

2. Statement of the problem of contact transport

Consider the system containing a large number of Q_i elements, which are distributed in space and moving in an arbitrary way. We shall call this system a background. It is assumed, by analogy with gases, that the average size of an element a is significantly less than the average distance d between them $d \gg a$. If N is the average density of such elements of the background, then a is proportional to $N^{-1/3}$ or $N^{-1/2}$ depending on whether the distribution of elements is spatial or planar. Moving elements are allowed to approach each other up to a distance ρ, which is less than the average distance between the elements, i.e. $d > \rho \gg a$. We shall call this single event a contact. Contacting elements can interact and keep close to each other for some long time or can pass by sufficiently close by each other.

Assume now that there are some of the elements of the background that may possess a specific property. We will call them carriers. It is assumed that this property can be transferred to the rest of the elements by the above defined contact with the carriers. It means that this property can propagate by means of contacts. We will call this process of propagation the contact transport.

The effectiveness of the contact transport depends upon the elements. Hence, ultimately it depends on the characteristics of a contact: the way it moves, how often it occurs, how long it lasts, etc. The contact parameters are defined by the background. Finally, the transfer of a property during contact is important. This transfer depends upon the distance between contacting elements, duration of the contact, the time interval between the contacts and the instant when the carrier first acquired some property, i.e. the 'memory' of the system. The last effect is influenced by the peculiarities of the transport of this certain property rather than by the parameters of the background.

It will be shown below that the transport process can be determined completely if the statistical characteristics of the background as well as the parameters of the contact transfer of some property are known.

3. The equations of contact transport

To formulate the equations of contact transport, let us assume that $n(\boldsymbol{r}, t, t_1)$ is the differential density of carriers. This means that the number of carriers in a unit of volume at the point $\boldsymbol{r}$, which acquired the certain property between the time t_1 and $t_1 + \mathrm{d}t_1$ $(t > t_1)$ is $n(\boldsymbol{r}, t, t_1)\,\mathrm{d}t_1$. The total density of carriers is given then by:

$$N(\boldsymbol{r}, t) = \int_0^t n(\boldsymbol{r}, t, t_1)\,\mathrm{d}t_1. \tag{3.1}$$

The introduction of the differential density indicates that the properties of elements of the gas-like system can vary in time, unlike the ordinary gases. Therefore the effectiveness of the contact transfer of the specified property might depend on the time interval $t-t_1$ following after the instant when the property has been first acquired by the carrier.

The equation of contact transport (ECT) is obtained for the differential density of carriers $n(\boldsymbol{r}, t, t_1)$. The variations of the differential density is due to the spatial transport of carriers as well as because of the transfer of a property to the elements of the background by contacts. The spontaneous losses can be taken into account by writing the ECT in the form:

$$\partial n/\partial t = S + J - Rn. \tag{3.2}$$

Here S is the operator of convective transport, J is the integral of contact transfer, and R is the function of spontaneous losses.

Let us discuss these expressions in detail. The operator of convective transport S is defined by the motion of the elements of background for which the statistic characteristics are assumed to be known. The operator S has the general form:

$$S = (1/\Delta t)\left\{\int K(\boldsymbol{r}, t; \boldsymbol{r}', t')n(\boldsymbol{r}', t', t_1)\,\mathrm{d}\boldsymbol{r}' - K_0(\boldsymbol{r}, t, t')n(\boldsymbol{r}, t, t_1)\right\} \tag{3.3}$$

where $K(\boldsymbol{r}, t; \boldsymbol{r}', t')$ is the correlation function which is essentially the probability of finding the element of the background at the moment t' at the point $\boldsymbol{r}'$ moving into point $\boldsymbol{r}$ at the moment t. The function $K_0(\boldsymbol{r}, t, t')$ is given by

$$K_0(\boldsymbol{r}, t, t') = \int K(\boldsymbol{r}, t; \boldsymbol{r}', t')\,\mathrm{d}\boldsymbol{r}' \tag{3.4a}$$

$$K(\boldsymbol{r}, t, \boldsymbol{r}', t') = K(\boldsymbol{r}', t', \boldsymbol{r}, t). \tag{3.4b}$$

According to equations (3.3) and (3.4) the operator S satisfies the conservation of the total number of elements:

$$\int S\,\mathrm{d}\boldsymbol{r} = 0. \tag{3.5}$$

The integral of contact transfer represents the number of new carriers appearing at the given moment t in a unit of time $\mathrm{d}t$ and volume $\mathrm{d}\boldsymbol{r}$ near the given point $\boldsymbol{r}$. This takes place due to transport to the elements of the background of the specific property by contacts with carriers. The integral of contact transfer is given by

$$\begin{aligned} J(\boldsymbol{r}, t, t_1) &= B(\boldsymbol{r}, t)\delta(t-t_1) \\ B(\boldsymbol{r}, t) &= \int_0^t \nu(\boldsymbol{r}, t, t-t_1)n(\boldsymbol{r}, t, t_1)\,\mathrm{d}t_1. \end{aligned} \tag{3.6}$$

Here the delta function δ, guarantees that the contact transfer takes place at the moment t. Therefore $t_1 = t$ for all new-born carriers. In (3.6) ν is the frequency of contact transfer, i.e. the number of contacts at the point $\boldsymbol{r}$, t produced by the carrier which acquired some property at time t_1. It is obvious that the value of ν is directly proportional to the density of elements of the background,

$$\nu(\boldsymbol{r}, t, t-t_1) = (N_0 - N)P(\boldsymbol{r}, t, t-t_1) \tag{3.7}$$

where N_0 is the density of background elements, N is the total density of carriers and $\bar{P}$ is the average efficiency of a contact transfer given by

$$\bar{P}(\boldsymbol{r}, t, t-t_1) = \int F(\boldsymbol{r}, t, \boldsymbol{q}) P(t-t_1, \boldsymbol{q})\, \mathrm{d}\boldsymbol{q} \tag{3.8}$$

where $\boldsymbol{q} = \boldsymbol{q}(q_1, q_2, \ldots)$ are the parameters of contact, such as its duration τ, minimum distance between the contacting elements ρ, the angle of contact Ω, which becomes important if the elements are anisotropic, etc. This defines F as a function of contact which shows the probability of contact at a given point $\boldsymbol{r}$, t having parameters $\boldsymbol{q}$. The function of contact F depends upon the properties of the background. This function is normalized in the usual way as,

$$\int F\, \mathrm{d}\boldsymbol{q} = 1. \tag{3.9}$$

The efficiency of the transfer of this certain property during contact with parameters $\boldsymbol{q}$ is described by the function $P(t-t_1, \boldsymbol{q})$. This shows the probability of transfer of this property with contact parameters $\boldsymbol{q}$ to an element of background while the carrier first acquired this property at time t_1. The efficiency P corresponds to the multiplication of the scalar velocity of molecule v by the cross section σ of interaction as is known from the dynamics of ordinary gases. Finally, $R(t-t_1)$ defines the average speed of spontaneous losses of a certain property by carriers.

The Cauchy problem for equation (3.2) is totally defined in the absence of boundaries by the given initial function

$$n(\boldsymbol{r}, t=0, t_1=0) = n_0(\boldsymbol{r}). \tag{3.10}$$

The solution of ECT can be found using the sum of the solutions of the linear equation

$$\partial n/\partial t = S - Rn \tag{3.11}$$

with initial functions $n(\boldsymbol{r}, t_1, t_1) = \bar{n}(\boldsymbol{r}, t_1)$ assigned at the instant $t = t_1$ by

$$\bar{n}(\boldsymbol{r}, t_1) = B(\boldsymbol{r}, t_1) = \int_0^{t_1} \nu(\boldsymbol{r}, t_1, t_1 - t') n(\boldsymbol{r}, t_1, t')\, \mathrm{d}t. \tag{3.12}$$

Here the function $n_0(\boldsymbol{r}) = n(\boldsymbol{r}, 0, 0)$ is defined by the initial condition (3.10), and t_1 is the continuous parameter of equations (3.11) and (3.12). These equations are equivalent to ECT (3.2) with the initial condition given by equation (3.10). The total density of carriers can be found using (3.11), (3.12):

$$\partial N/\partial t = S(N) + \int_0^t (\nu - R) n(\boldsymbol{r}, t, t_1)\, \mathrm{d}t_1. \tag{3.13}$$

Here $S(N) = \int_0^t S(n)\, \mathrm{d}t_1$, which can be proved by integrating both sides of (3.3) over $\mathrm{d}t_1$.

It should be noted that equation (3.2) can be easily extended to describe the transport in the non-uniform background consisting of groups (or layers) having different distributions and functions of contacts. The ECT which accounts for the above different groups are written for each group of carriers $n_k(\boldsymbol{r}, t, t_1)$:

$$\partial n_k/\partial t = S_k + \sum_{k'} B_{kk'} \delta(t-t_1) - R_k n_k \tag{3.14}$$

where the operators of transport S_k and the integrals of contact $B_{kk'}$ can be described by (3.3)-(3.7) replacing n with n_k, and $N_0 - N$ with $N_{0k} - N_k$. The solution of the Cauchy problem for the system of equations (3.14) can be found in the same way as for (3.11), (3.12). If the group distribution is continuous, the integration over the index k is assumed in (3.14) instead of summation.

Finally, it should be noted that the operator of convective transport can be written in the form of divergence if the dispersion of carriers velocities is not too large:

$$S = \partial j / \partial r$$

$$j = (1/\Delta t) \int (r' - r)\{K(r, t; r', t - \Delta t) n(r', t - \Delta t, t_1) - K(r', t - \Delta t; r, t) n(r, t, t_1)\}\, dr'. \tag{3.15}$$

The flux j can be simplified if the correlation function (3.4) is local, compared to the distribution of carriers: $(\Delta r / n)|\partial n / \partial r| \ll 1$, where Δr is the characteristic spatial width of the correlation function, assuming that it decreases rapidly if $|r' - r| \gg \Delta r$. Under these conditions the flux j acquires the simpler form:

$$j_k = V_k n - D_{ik} \partial n / \partial r_i$$

$$V = (1/\Delta t) \int (r - r') K(r', t - \Delta t; r, t)\, dr' \tag{3.16}$$

$$D_{ik} = (0.5/\Delta t) \int (r - r')_i (r - r')_k K(r', t - \Delta t; r, t)\, dr'$$

where V is the average directed velocity of elements of the background and D_{ik} are the components of the diffusion tensor. General results obtained are further analysed below.

4. The nonlinear waves

Let us start with the assumption that the average effectiveness of the contact transfer given in (3.8) is constant, i.e.

$$P(r, t, t - t_1) = \text{constant} = P_0. \tag{4.1}$$

Under this condition the integral of contact transfer (3.12) acquires a simple form:

$$B(r, t) = P_0(N_0 - N) \int_0^t n(r, t, t_1)\, dt_1 = P_0(N_0 - N) N. \tag{4.2}$$

In this case the differential density n plays no role because the contact transfer does not depend on the moment t_1 when the carrier was activated, i.e. system has no memory. Therefore we can pass to the total density of carriers in the ECT by integrating over t_1. Assuming, in addition, that the function of spontaneous losses is constant $R = T_0^{-1}$, where T_0 is the lifetime of a carrier, we obtain using equations (3.2), (3.13) and (4.2) the following result:

$$\partial N / \partial t = S(N) + P_0(N - N_0) N - N / T_0. \tag{4.3}$$

Let us discuss the characteristic solutions of the transport equation (4.3). For the sake of simplicity, the operator of transport S will be taken in the diffusion form (3.16) with $V = 0$. It is convenient to introduce the dimensionless variables as follows:

$$n = N/[N_0(1-T^{-1})] \qquad \tau = t\nu_0(1-T^{-1})$$
$$\nu_0 = P_0 N_0 \qquad T = T_0\nu_0. \tag{4.4}$$

In the one-dimensional case it can be found that

$$\partial n/\partial\tau = \partial^2 n/\partial\xi^2 + n(1-n)$$
$$\xi = x/[D/(\nu_0(1-T^{-1}))]^{0.5}. \tag{4.5}$$

From (4.5) it can be seen that the system is bistable; it possesses two uniform steady states: we call them the ground state $n = 0$, and the excited state $n = 1$. In the ground state there are no carriers at all. The concentration of carriers in the excited state, existing under condition $T > 1$, is significant however. The transition to the excited state out of the ground state is described by the following equation (for the uniform condition $n_0(x) = \text{constant} = \varepsilon$):

$$dn/d\tau = n(1-n) \qquad n(0) = \varepsilon. \tag{4.6}$$

The solution of the above equation is:

$$n = \varepsilon/[\varepsilon + (1-\varepsilon)\exp(-\tau)]. \tag{4.7}$$

It can be easily shown now, that the ground state under the condition $T > 1$, i.e.

$$\nu_0 > 1/T_0 \tag{4.8}$$

is unstable. It means that the appearance of a small amount of carriers ε leads to the exponential increase of their number in time until the system reaches an excited uniform state $n = 1$. This state is stable under condition (4.8).

4.1. The structure of the moving front

A steady spatial front for the transition between the ground and the excited state does not exist. Nevertheless, a front moving with a constant velocity V is possible, which is similar to the effect of shock waves in ordinary gas dynamics [12]. Let us discuss the structure of this front. Assuming in (4.5) that,

$$n = n(z) \qquad z = \xi - V\tau \qquad n(z\to-\infty)\to 1 \qquad n(z\to+\infty)\to 0 \tag{4.9}$$

we can then obtain

$$d^2n/dz^2 = -V\,dn/dz - n(1-n) \tag{4.10}$$

or, equivalently,

$$dp_1/dn = -V - n(1-n)/p_1 \qquad p_1 = dn/dz. \tag{4.11}$$

The boundary conditions for (4.11) can be found from (4.9) as

$$p_1 \to 0 \text{ under } n\to 0 \qquad p_1 \to 0 \text{ under } n \to 1. \tag{4.12}$$

Values $n = 0$ and $n = 1$ create singularities. Expanding p_1 in the vicinity of singularities $p_1 = -k_0 n + \ldots$; $p_1 = k_1(n-1) + \ldots$, we obtain the values of k_0 and k_1 given by:

$$k_0 = V/2 + (V^2/4 - 1)^{1/2} \qquad k_1 = -V/2 + (V^2/4 + 1)^{1/2}. \tag{4.13}$$

It follows from this expression that the physically meaningful solution exists under the condition:

$$V > V_c = 2. \tag{4.14}$$

Expanding p_1 in power series of $1/V^2$ we obtain

$$p_1 = -(n(1-n)/V)[1+(1-2n)/V^2+2(1-5n+5n^2)/V^4 + (5-46n+108n^2-72n^3)/V^6+\ldots]. \tag{4.15}$$

The above solution satisfies the boundary condition (4.12). It can be obtained using (4.9), (4.10) and (4.15) that for $V\to\infty$ equation (4.9) acquires the following form:

$$n = (1+\exp(z/V))^{-1} \qquad p_1 = -n(1-n)/V. \tag{4.16}$$

Figure 1 demonstrates the dependence of p_1 on n for the different values of V. In accordance with condition (4.14), the solution of (4.10) satisfying the boundary conditions (4.12) exists only for $V>2$. The structure of the moving front is always close to that determined by (4.16). The width of the front increases directly proportional to V. In the original variables the characteristic width and velocity of the front are given by:

$$L = V[D/(P_0N_0(1-1/T))]^{1/2}$$
$$v = (V/2)v_c \qquad v_c = 2[DN_0P_0(1-1/T)]^{1/2}. \tag{4.17}$$

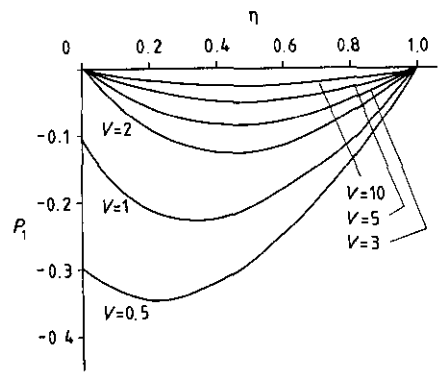

Figure 1. Parameter $p_1 = dn/dz$ as a function of dimensionless concentration of carriers n. The different curves have different values of dimensionless velocity V of the transition front between the ground and the excited states.

4.2. The evolution of the local initial disturbance

It was shown above that the ground state, in which carriers are absent, $n = 0$, is unstable for $\nu_0 > T_0$. We would like now to discuss the time evolution of a small disturbance of this state localized in the vicinity of a certain point $\xi = 0$ at the moment $\tau = 0$, i.e.

$$n_1(\xi, 0) = n_{10}(\xi) \qquad \int n_{10}(\xi)\,d\xi = N_0 \qquad N_0 \ll 1. \tag{4.18}$$

The time evolution of n_1 is described by the linearized form of (4.5):

$$\partial n_1/\partial\tau = \partial^2 n_1/\partial\xi^2 + n_1. \tag{4.19}$$

The solution of this equation satisfying the initial condition (4.18) is given by:

$$n_1 = N_0/2\sqrt{\pi\tau}\,\exp\{-\xi^2/4\tau+\tau\}. \tag{4.20}$$

Using this expression we can find that the number of carriers rapidly increases in time in the region $\xi<2\sqrt{\tau}$. The linear approximation fails under these circumstances. Actually, the value of n_1 tends to 1 in the region described by the above inequality under the condition:

$$\tau > \tau_c = \ln[2N_0^{-1}\sqrt{\pi\ln(N_0^{-1})}]. \tag{4.21}$$

In this case the nonlinear term determining the excited steady state becomes essential. Linearizing the equation (4.5) in the vicinity of the excited state, we also obtain

$$\partial n_2/\partial\tau = \partial^2 n_2/\partial\xi^2 - n_2 \qquad n_2 = 1-n. \tag{4.22}$$

The solution of this equation is given by:

$$n_2 = N_2/2\sqrt{\pi\tau}\,\exp\{\xi^2/4\tau-\tau\}. \tag{4.23}$$

In view of this, the approximate solution of (4.5) satisfying the initial condition (4.18) has the form

$$n = \begin{cases} N_0/2\sqrt{\pi\tau}\,\exp\{-\xi^2/4\tau+\tau\} & \text{for } n<0.5 \\ 1-N_2/2\sqrt{\pi\tau}\,\exp\{\xi^2/4\tau-\tau\} & \text{for } n>0.5. \end{cases} \tag{4.24}$$

The value of N_2 can be found taking into account the continuity of solution n under the $\tau=\tau_c$ which is determined by the expression (4.21)

$$N_2 \simeq N_0\exp(2\tau_c) \simeq 4\pi N_0^{-1}\ln(N_0^{-1}). \tag{4.25}$$

Figure 2 illustrates the results given by (4.24) and (4.25). It can be seen, that the region $|\xi|<\xi_f$ occupied by excited state $n=1$ expands rapidly in time. We find that, approximately, in this region

$$\xi_f = 2\tau[1-\ln(\sqrt{\pi\tau}/N_0)/2\tau]. \tag{4.26}$$

Whence, the front of the region occupied by carriers moves with the velocity close to critical v_c (see (4.17)). It should be underlined that a strictly steady front does not exist. The structure as well as the velocity of the front varies slowly in time.

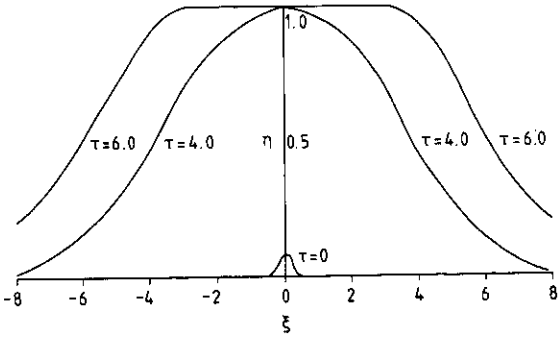

Figure 2. The dependence of the concentration of the carriers n on coordinate ξ for the different values of time τ.

4.3. The wave motions

Consider now the transport including the carriers N as well as the elements of background $N_1 = N_0 - N$. In this case (3.14), (3.16) and (4.5) have the form:

$$\begin{aligned} &\partial n/\partial t = -V\partial n/\partial x + D\partial^2 n/\partial x^2 + \nu_0 n n_1 - n/T_1 \\ &\partial n_1/\partial t = -V_1\partial n_1/\partial x + D\partial^2 n_1/\partial x^2 - \nu_0 n n_1 + n_1/T_0 \\ &n = N/N_0 \qquad n_1 = N_1/N_0. \end{aligned} \tag{4.27}$$

In contrast to (4.5), the average transport velocities V and V_1 are taken into consideration. The system of equations (4.27) possesses two steady states, as discussed before.

$$n = 0 \qquad n_1 = 0 \qquad n_0 = 1/T_0\nu_0 \qquad n_{10} = 1 - 1/T_0\nu_0 = 1 - n_0. \tag{4.28}$$

Taking into account that in the steady state $n_1 = 1 - n$, we find that

$$T_1 = T_0(\nu_0 T_0 - 1)^{-1}. \tag{4.29}$$

Let us discuss weak disturbances of the excited steady state. Representing the disturbances in the Fourier form

$$\begin{aligned} &n = n_0 + n' \exp(-i\omega t + ikx) \\ &n_1 = (1 - n_0) + n_1' \exp(-i\omega t + ikx). \end{aligned} \tag{4.30}$$

Let us substitute these expressions into (4.27) we can obtain that

$$\begin{aligned} &-i\omega n' = -iVkn' - Dk^2 n' + \nu_0 n_0 n_1' \\ &-i\omega n_1' = -iV_1 k n_1' - Dk^2 n_1' - \nu_0(1 - n_0)n' \end{aligned} \tag{4.31}$$

which gives us the following dispersion relations:

$$\omega_{1,2} = k(V + V_1)/2 + \{k^2(V - V_1)^2/4 + \nu_0^2 n_0(1 - n_0)\}^{0.5} - iDk^2. \tag{4.32}$$

From (4.32) it can be seen that two wavemodes can propagate in the system. The dispersion relation for each of them depends on the difference of transport velocities of carriers and elements of background. As in ordinary hydrodynamics [12], the wave fading is determined by diffusion.

It should be noticed that the dispersion relations will become degenerate if the variation in the number of elements of the background is caused only by the transition of carriers due to their spontaneous losses. In this case the last term in the second of equations (4.27) has the form of n/T_1, and the dispersion relations will change to

$$\omega_1 = kV - ik^2 D \qquad \omega_2 = kV_1 - i(n_0\nu_0 + k^2 D). \tag{4.33}$$

5. Instability of the ground state. Self-excitation of oscillations

In the previous section, the time variation of the efficiency of contact transfer has been neglected. This approximation allows the use of the simplified equation (4.3) describing the total density of carriers. Let us discuss a more general case when only the dependence P on time is sufficient. We start by investigating the stability of the ground state $n = 0$. For this purpose, the linearized equations (3.11), (3.12) and the transport operator in the diffusion form (3.16) are going to be used. The linearization means that the function of spontaneous losses depends only on time, and the diffusion coefficient is taken to

be constant. Also the average directed velocity of elements of the background is taken to be equal to zero for the sake of simplicity. In terms of Fourier components over r we obtain

$$\partial n_k/\partial t = -Dk^2 n_k - R(t-t_1)n_k \tag{5.1}$$

$$\bar{n}_k(t_1) = \int_0^{t_1} \nu(t_1-t)n_k(t_1, t)\,dt \tag{5.2}$$

where $\nu(\tau)$ is the linearized frequency of the contact transfer, $R(\tau)$ is the function of spontaneous losses which is determined by the experimental conditions. The solution of (5.1) is given by:

$$n_k(t, t_1) = \bar{n}_k(t_1)\exp\left\{-Dk^2(t-t_1) - \int_0^{t-t_1} R(\tau)\,d\tau\right\}. \tag{5.3}$$

The asymptotic solution of (5.2) can be determined by substituting expression (5.3) into (5.2). It is given by

$$\bar{n}_k(t_1) = \bar{n}_k(0)\exp(pt_1) \tag{5.4}$$

where p is the largest root of the equation:

$$1 = \nu^*(p)$$
$$\nu^*(p) = \int_0^{\infty} \exp\{-p\tau\}\nu(\tau)\exp\left\{-Dk^2\tau - \int_0^{\tau} R(\tau_1)\,d\tau_1\right\}d\tau. \tag{5.5}$$

In fact, within the approximations made, (5.5) takes the form

$$1 = \nu^*(p) \qquad p^* = p + 1/T_0 + Dk^2 \tag{5.6}$$

where $\nu(p^*)$ is the Laplace transform of $\nu(\tau)$. Equation (5.6) was obtained under condition $R(\tau) = \text{constant} = 1/T_0$.

5.1. Examples

Under the constant values of $\nu = \nu_0$ and $R = 1/T_0$ the value $\nu^*(p)$ is $\nu^*(p) = \nu_0/p^*$, and using (5.6) we obtain

$$p = \nu_0 - 1/T_0 - Dk^2. \tag{5.7}$$

In this case the instability of the ground state appears if $\nu_0 > 1/T_0$ for the spatially uniform distribution ($k \to 0$) which is in agreement with condition (4.8).

If the frequency $\nu(\tau)$ can be represented as a finite pulse

$$\nu(\tau) = \begin{cases} \nu_0 & \text{if } \tau_0 < \tau < \tau_1 \\ 0 & \text{if } \tau < \tau_0,\ \tau > \tau_1 \end{cases}$$

the equation (5.6) (under $k \to 0$) acquires the form

$$\nu_0\{\exp(-p^*\tau_0) - \exp(-p^*\tau_1)\} = p^*$$
$$p^* = p + 1/T_0 + Dk^2. \tag{5.8}$$

The condition:

$$\nu_0(\tau_1 - \tau_0) > 1 \tag{5.9}$$

determines the physical root of equation (5.8). The instability will occur if $p>0$ in addition to the condition (5.9). This means that the root of the equation (5.8) should be larger than $1/T_0+Dk^2$. This condition is always satisfied when the values of T_0 are large enough and k tends to zero.

Another case of significant interest is when the function $\nu(\tau)$ has a sharp maximum close to some prescribed value of τ_m. In this case we obtain

$$\nu(p)=K_0\exp(-p\tau_m) \qquad K_0=\int_0^\infty \nu(\tau)\,d\tau \tag{5.10}$$

and (5.6) acquires a simple solution

$$p=-1/T_0-Dk^2+(1/\tau_m)\ln K_0. \tag{5.11}$$

From this equation it can be seen, that the instability rises if $K_0>\exp(\tau_m/T_0)$. The inequality $K_0>1$ is the necessary condition for instability, which is in agreement with the condition (5.9).

The presence of instability in the ground state does not guarantee that the steady excited state exists. It shows only that the transition to the oscillating regime is possible. The equations (5.1) and (5.2) are related to models of spread of a disease which does not induce permanent immunity [13]. The existence of periodic solutions of this equation was established in [14]. Oscillations always arise if the ground state is unstable and the integral of the frequency of contact transport over time is bounded if the integral of the function of spontaneous losses over time tends to infinity. The function $R(\tau)$ has been defined above;

$$\int_0^\infty \nu(\tau)\,d\tau=K_0 \qquad \int_0^\infty R(\tau)\,d\tau\to\infty. \tag{5.12}$$

For instance, let us discuss the situation when the function $\nu(\tau)$ described by (5.10), has a sharp peak. The nonlinear equations for the density of carriers then can be obtained using (3.2), (5.1) and (5.2) under assumption that the spontaneous losses are constant and that the transport is diffusional:

$$\begin{aligned} &n_k(t)=K_0(1-N_k(t))n_k(t-\tau_m)\exp(-p_0\tau_m)\\ &p_0=1/T_0+Dk^2\\ &N_k(t)=\exp(-p_0t)\int_0^t n_k(t_1)\exp(p_0t_1)\,dt_1. \end{aligned} \tag{5.13}$$

The stability of the ground state for this case has been previously investigated (see (5.11)). The excited steady state $N=$ *constant does not exist, as shown in* (5.13). This means that the system undergoes a transition to an unstable regime under the conditions of instability discussed after (5.11). Figure 3 shows the oscillating solutions of (5.13) (when $k\to 0$), found by computer simulations.

It should be noted that the development of a weak initial disturbance given by (4.18) occurs in the form of the nonlinear wave diverging on both sides with velocity $v\simeq 2(Dp)^{1/2}$. Where p is the increment of instability of the ground state defined by the expression (5.11). The motion of fronts is accompanied now by the wave excitation with the wavelength $\lambda\simeq vT$. Hence, the instability of the ground state (5.5) in the discussed system leads to the self-excitation of waves and oscillations under the conditions (5.12).

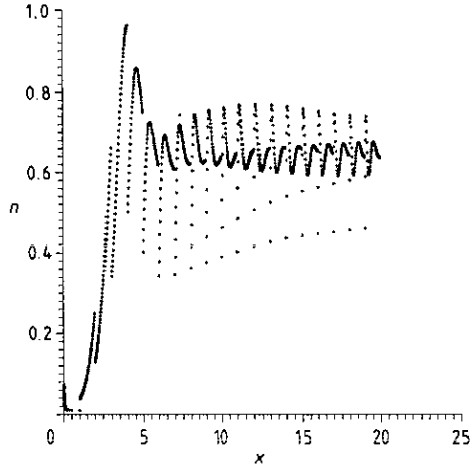

Figure 3. The time oscillations of the density of carriers n with the latent period τ_m; $x = t/\tau_m$. It was assumed that $T_0 = \tau_m = 2$, and $K_0 = e^2$. The initial concentration of carriers was taken $10^{-2}\exp\{(\ln K_0 - \tau_m/T_0)x\}$, in accordance with (5.11).

6. The strongly non-uniform system

Consider now the case of the strongly non-uniform medium. Let us assume that the elements of the system are localized close to specified points $\boldsymbol{r}_k$

$$N_0(\boldsymbol{r}) = \sum_k n_{k0}\delta(\boldsymbol{r} - \boldsymbol{r}_k). \tag{6.1}$$

These points play the role of sources in the transport process. If the density of elements is very high inside the sources then the contact transport goes fast inside the source which results in the generation of a large number of carriers. However, propagation proceeds slowly between the sources. In this case the contact transport occurs in a similar manner to the flux of the disturbance between the sources breaking out one after another. This situation is close to that which is discussed in percolation theory [8–10]. In addition the given approach considers only equations for mean density and ignores fluctuations, which become important near threshold, as it has been shown in [8–10].

Assuming that condition (4.1) is fulfilled, the equation of contact transport (4.3) can be simplified. Taking into account the discreet character of sources, the equation (4.3) can be rewritten in the integral form

$$N(\boldsymbol{r}, t) = p_0 \int \mathrm{d}\boldsymbol{r}' \int_0^t G(\boldsymbol{r} - \boldsymbol{r}', t - t')N(\boldsymbol{r}', t')(N_0 - N(\boldsymbol{r}', t'))\,\mathrm{d}t' \tag{6.2}$$

where $G(\boldsymbol{\rho}, t)$ is the Green function of the corresponding linear problem (without the integral of contact transfer). For example, the Green function for the diffusion transport on the planar surface has the same behaviour as in directed percolation [15].

$$G(\boldsymbol{\rho}, t) = (4\pi Dt)^{-1}\exp(-\rho^2/4Dt - t/T_0). \tag{6.3}$$

The contact transport occurs infinitely fast, like an outbreak, inside any source because, according to (6.1), the density of elements is infinitely large inside it. Assuming that the carriers appear at the moment t_k in the source localized at the point $\boldsymbol{r}_k$, the solution

of (6.2), (6.3) can be expressed as

$$N(\boldsymbol{r}, t) = (4\pi D)^{-1} \sum_k J(t-t_k)/(t-t_k) \exp\{-(\boldsymbol{r}-\boldsymbol{r}_k)^2/(4D(t-t_k)) - (t-t_k)/T_0\} \quad (6.4)$$

where

$$J(t-t_k) = \begin{cases} 0 & \text{if } t-t_k < 0 \\ n_{k0} & \text{if } t-t_k > 0. \end{cases} \quad (6.5)$$

Here n_{k0} is the total number of elements in a given source. They become carriers at the moment $t = t_k$.

The ignition time of each outbreak can be also defined. We had assumed, that in the initial moment the localized disturbance (4.18) is given. The density of carriers immediately becomes different from zero on the whole surface due to the diffusion transport. If we assume that the density of elements is infinite in any source, as in (6.1), then all sources break out at once at the time $t = 0$. However, if a finite density of carriers is required to excite a source $N(\boldsymbol{r}_k, t) = \varepsilon_k$, the condition of excitation acquires the form:

$$N(\boldsymbol{r}_k, t) = (4\pi D)^{-1} \sum_k J_n(t-t_n)/(t-t_n) \exp\{-\rho_{nk}^2/(4D(t-t_n)) - (t-t_n)/T_0\} > \varepsilon_k \quad (6.6)$$

where $\rho_{nk} = |\boldsymbol{r}_k - \boldsymbol{r}_n|$ and $k \neq n$.

Condition (6.6) determines the ignition time of the source of number k. The value of $N(\boldsymbol{r}_k)$ decreases exponentially when ρ_{nk} increases. It shows, that the main role in excitation is played by the source's closest neighbours. The sharp dependence on time of the expression in the exponent in (6.6) can be used to simplify the results. The extremum value of time is given by

$$t_m = t_n + \rho_{nk}(T_0/4D)^{1/2}. \quad (6.7)$$

We also can determine the optimal input into the point $\boldsymbol{r}_k$ by the source located at the point $\boldsymbol{r}_n$:

$$(\Delta N)_{nk} = n_{n0}/[2\pi\rho_{nk}(DT_0)^{1/2}] \exp\{-\rho_{nk}/(DT_0)^{1/2}\}. \quad (6.8)$$

The result (6.8) suggests, that if on average

$$\rho_{nk}^2 \gg DT_0 \quad (6.9)$$

the opportunity of the source $\boldsymbol{r}_k$ to outbreak depends actually upon the closest source $\boldsymbol{r}_{k-1}$. Sources will outbreak one after the other and the ignition time of the source $\boldsymbol{r}_k$ is:

$$t_k = t_{k-1} + 0.5\,T_0\{\ln[n_{k-1,0}/(2\pi DT_0\varepsilon_{k0})] + [\ln^2(n_{k-1,0}/(2\pi DT_0\varepsilon_{k0})) - \rho_{k-1,k}^2/DT_0]^{1/2}\} \quad (6.10)$$

where t_{k-1} is the ignition time of the $k-1$ source, and $n_{k-1,0}$ is the total number of its elements. It should be emphasized that the outbreak takes place only if $(\Delta N)_{k-1,k} > \varepsilon_{k0}$, i.e. if the expression under the square root in (6.10) is positive. In the opposite case the outbreak of the kth source will not occur. Under this condition the contact transport is similar to the 'flowing' of disturbances through the localization points $\boldsymbol{r}_{k-1}$.

Hence, under condition (6.9) the process of propagation of initial disturbance in the system is non-trivial: it strongly depends on the spatial distribution of localization

points. If the condition opposite to (6.9) is achieved, the system is close enough to the uniform.

7. Discussion

Generalized equations have been obtained describing contact transport in a system of free moving elements interacting in a prescribed way. The contact parameters, the internal properties and the distribution of elements can be some prescribed functions of time. Possible applications of the above transport processes in gas-like systems are associated with the mathematical theory of contagious diseases. The systems described by these equations include the carriers of a certain property and the background elements. Such systems turn out to be bistable; possessing two steady states: ground and excited. The structure of the moving front formed by the transition between ground and excited states has been discussed. These results can be used to evaluate the velocity of propagation of an epidemic disease.

It has been shown, that for a constant value of an average effectiveness of contact transfer, i.e. in the system without memory, two different modes can propagate. Their dispersion relations depend on the transport velocities of both carriers and on the background elements.

The stability of the ground state has been investigated using the time-dependent value of the average effectiveness of contact transfer. The instability of the ground state leads to self-excitation of waves and oscillations which is similar to time oscillations known in the epidemic processes [1].

Contact transport in highly non-uniform media consisting of a number of point sources has been studied as well. The ignition time of a source has been determined. Also a criterion has been obtained which gives conditions under which the medium becomes non-uniform. The contact transport in highly non-uniform media can be associated with an outbreak of contagious disease in the region of large concentration of susceptibles.

Acknowledgments

The authors express their appreciation to L P Pitaevskii, E S Kolayeva, A P Mesherkin for their comments and advice. Special thanks are given to A Kholodenko for reading the draft of this manuscript and providing some helpful comments.

References

[1] Bailey N T J 1975 *The Mathematical Theory of Infectious Diseases and its Application* (London: Griffin)
[2] Anderson R M 1982 *The Population Dynamics of Infectious Diseases: Theory and Applications* (London: Chapman and Hall)
[3] Hyman J M and Stanley E A 1988 *Math. Biosci.* **90** 415–73
[4] Hethcote H W and Van Ark J W 1987 *Math. Biosci.* **84** 85–118
[5] Dietz K 1988 *Math. Biosci.* **90** 397–412
[6] Longini I M Jr 1988 *Math. Biosci.* **90** 367–83
[7] Rvachev L A and Longini I M Jr 1985 *Math. Biosci.* **75** 3–22
[8] Grassenberger P and De La Torre A 1979 *Ann. Phys.* **122** 373–96

[9] Grassenberger P 1989 *J. Phys. A: Math. Gen.* **22** L1103–7
[10] Cardy J L 1983 *J. Phys. A: Math. Gen.* **16** 709–12
[11] Marques M C 1990 *Physica* **163A** 915–31
[12] Landau L D and Lifshitz E M 1959 *Fluid Mechanics* (London: Pergamon)
[13] Hethcote H W, Stech H W and van den Driessche P 1981 *Differential Equations and Applications in Ecology, Epidemics and Population Problems* ed S N Busenberg and K L Cooke (New York: Academic) pp 65–82
[14] Gripenberg G 1980 *J. Math. Biol.* **10** 271–80
[15] Cardy J L and Sugar R L 1980 *J. Phys. A: Math. Gen.* **13** L423–7